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On Bounded, Periodic, and Almost Periodic Solutions for a System of Nonlinear Second-Order Ordinary Differential Equations

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1. INTRODUCTION

Let us consider a system of second-order ordinary differential equations of the form

$$\ddot{x} + \rho(t, x, \dot{x}) + \beta(x) = f(t), \qquad (\cdot = d/dt), \tag{1}$$

where $x(t)$, ρ, β, f are all m-dimensional vector valued functions, i.e., $x(t) = (x^1(t), x^2(t),..., x^m(t))^T$, $\rho(t, x, x) = (\rho^1(t, x, x),..., \rho^m(t, x, x)^T$, etc.

The object of this paper is to prove existence theorems concerning bounded, periodic, and almost periodic solutions of system (1) and investigate some of their asymptotic properties.

Since the special case $\rho(t, x, \dot{x}) = \rho(\dot{x})$ and $\beta(x) = B \cdot x$, where B is a positive matrix, was investigated by Caccioppoli and Ghizzetti [3] and Amerio [1], those solutions of the second-order equations have been treated by many authors. In particular we mention the lecture notes [5] by Fink, the book [9] by Yoshizawa and the references cited there. Nevertheless our method and result here seem to be new and different from those of earlier works. Our method and result are related to those of Amerio and Prouse [2, part II] and Nakao [6, 7, 8], who considered nonlinear hyperbolic partial differential equations.

2. PRELIMINARIES AND RESULTS

For $u \in R^m$, where m is a positive integer, we denote the usual Euclidean norm by $|u|$ and the inner product by $(\cdot, \cdot)$. The functions to be considered are all real valued. Since we employ standard notations for function spaces, precise definitions of them will be omitted.

First we state our hypotheses on ρ and β. Among them, H_1 and H_2 are made for the existence of bounded solution, and H_3 and H_4 are for the uniqueness of it.

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H_1. $\rho(t, u, v)$ is defined and continuous on a domain $B_{r_0,r_1} = \{(t, u, v) \in R \times R^m \times R^m \mid |u| \leqslant r_0 \text{ and } |v| \leqslant r_1\}$ with some positive constants r_0, r_1, and satisfies

$$K_0(r_0, r_1) \mid v \mid^{p_0+2} \leqslant (\rho(t, u, v), v) \quad \text{and} \quad \mid \rho(t, u, v) \mid \leqslant K_1(r_0, r_1) \mid v \mid \tag{2}$$

if $|u| \leqslant r_0$, $|u| \leqslant r_1$, where p_0 is a nonnegative number, K_0, K_1 are positive constants depending on r_0 and r_1.

H_2. $\beta(x)$ is defined and continuous on $B_{r_0} = \{x \in R^m \mid |x| \leqslant r_0\}$ and satisfies

$$K_2(r_0) \mid x \mid^{p_1+2} \leqslant (\beta(x), x) \qquad \text{if} \quad |x| \leqslant r_0, \tag{3}$$

for some $p_1 \geqslant 0$ and $K_2(r_0) > 0$. Moreover, we assume, $\beta(x)$ has the form $\beta(x) = B \cdot x + \tilde{\beta}(x)$, where B is a nonnegative symmetric $m \times m$ matrix and $\tilde{\beta}(x)$ is of the form: $\tilde{\beta}(x) = (\tilde{\beta}^1(x^1), \ldots, \tilde{\beta}^m(x^m))^T$.

H_3. $\rho(t, u, v)$ is defined and continuous on B_{r_0,r_1}, and satisfies

$$K_0(r_0, r_1) \mid v_1 - v_2 \mid^2 \leqslant (\rho(t, u, v_1) - \rho(t, u, v_2), v_1 - v_2)$$

and (4)

$$\mid \rho(t, u_1, v_1) - \rho(t, u_2, v_2) \mid \leqslant K_2(r_0, r_1) \mid v_1 - v_2 \mid + K_3(a, b) \mid u_1 - u_2 \mid$$

if $|u| \leqslant a \leqslant r_0$ and $|v| \leqslant b \leqslant r_1$ for some K_0, $K_2 > 0$ and $K_3(a, b) \geqslant 0$, $K_3(a, b)$ tending to 0 as $a + b \to 0$.

H_4. As in H_2, $\beta(x)$ has the form $\beta(x) = B \cdot x + \tilde{\beta}(x)$ and moreover

$$C_0 \mid x \mid^2 \leqslant (B \cdot x, x) \leqslant C_1 \mid x \mid^2 \tag{5}$$

for some positive constants C_0, C_1, and

$$\mid \tilde{\beta}(x) - \tilde{\beta}(y) \mid \leqslant K_4(a) \mid x - y \mid \tag{6}$$

if $|x|$, $|y| \leqslant a \leqslant r_0$, where $K_4(a)$ is nonnegative constant tending to 0 as $a \to 0$.

Finally, regarding $f(t)$, we assume:

H_5. $f(\cdot) \in L_{\mathrm{loc}}^{(p_0+2)/(p_0+1)}(R)$ and

$$M \equiv \sup_{t \in R} \left(\int_t^{t+1} \mid f(s) \mid^{(p_0+2)/(p_0+1)} ds \right)^{(p_0+1)/(p_0+2)} < +\infty. \tag{7}$$

Next, we state our definitions of solutions for (1).

DEFINITION 1. $x(t)$ is said to be a bounded solution of (1) if $x(t) \in C^1(R)$ and the following conditions are satisfied:

$$x(t_2) - x(t_1) + \int_{t_1}^{t_2} \{\rho(t, x(t), \dot{x}(t)) + \beta(x(t)) - f(t)\}\, dt = 0 \tag{8}$$

for $\forall t_2 \geqslant \forall t_1$, and

$$\sup_{t \in R}(|x(t)| + |\dot{x}(t)|) < +\infty. \tag{9}$$

DEFINITION 2. $x(t)$ is said to be a bounded solution of (1) on $[0, T)$ $(0 < T \leqslant \infty)$ with initial condition (x_0, x_1) if $x(t) \in C^1([0, T))$, $x(0) = x_0$, $\dot{x}(0) = x_1$, and (8) is valid for $0 \leqslant t_1 \leqslant t_2 < T$, and moreover, $\sup_{t \in [0,T)}(|x(t)| + |\dot{x}(t)|) < +\infty$.

To state our results we introduce some notation. Let

$$J_0(x) = \tfrac{1}{2}(B \cdot x, x) + \sum_{i=1}^{m} \int_0^{x^i} \tilde{\beta}^i(s)\, ds \qquad \text{and} \qquad J_1(x) = (\beta(x), x)$$

for $x \in R^m$ with $|x| \leqslant r_0$. Then for each $x(\neq 0) \in R^m$, $\lambda_0(x)$ is defined as the minimum maximal point of $J_0(\lambda(x/|x|))$ on $0 \leqslant \lambda \leqslant r_0$ (note that $\lambda_0(x)$ may be equal to r_0). We put $\overline{r_0} = \min_{x \in R^m}(\lambda_0(x))$ and $D_0 = \min(\min_{x \in R^m} J_0(\lambda_0(x)(x/|x|)), \frac{1}{2}r_1{}^2)$. It is easy, by assumption H_2, to see that the number $\overline{r_0}$ and D_0 are positive.

We also introduce an energy function $V(x(t))$ and the stable set W as follows.

$$V(x(t)) = \tfrac{1}{2}|\dot{x}(t)|^2 + J_0(x(t)).$$

$$W = \{(x_0, x_1) \in R^m \times R^m \mid \|(x_0, x_1)\|_V \equiv \tfrac{1}{2}|x_1|^2 + J_0(x_0) < D_0$$

$$\text{and} \quad |x_0| < \overline{r_0}\}.$$

Now, we are ready to state our main results.

THEOREM 1. *Let $(x_0, x_1) \in W$. Then under assumptions H_1, H_2, and H_5, there exists a constant $M_0 \equiv M_0(D_0 - \|(x_0, x_1)\|_V)$ such that if $M < M_0$ the system (1) admits a bounded solution $x(t)$ on $[0, \infty)$ satisfying initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_1$. In particular we have, for $\forall t \in R^+ \equiv [0, \infty)$,*

$$|\dot{x}(t)| < (2D_0)^{1/2} \qquad \textit{and} \qquad |x(t)| < \overline{r_0}. \tag{10}$$

THEOREM 2. *Under assumptions H_1, H_2, and H_5, there exists a constant $\overline{M_0}$ $(\equiv M_0(D_0))$ such that if $M < \overline{M_0}$ problem (1) admits a bounded solution $x(t)$ on R, and we have for $\forall A \in R$*

$$|\dot{x}(t)| \leqslant b(M) \leqslant 2(D_0)^{1/2} \qquad \textit{and} \qquad |x(t)| \leqslant a(M) \leqslant \overline{r_0}, \tag{11}$$

where $a(M)$ and $b(M)$ are constants depending on M, tending to 0 as $M \to 0$.

THEOREM 3. *Under assumptions H_3, H_4, there exists a pair of positive numbers (a_0, b_0) such that for $a < a_0$, $b < b_0$, (1) has at most one bounded solution $x(t)$ in the domain $B^1_{a,b} = \{x(t) \in C^1(R) \mid |x(t)| \leqslant a, |\dot{x}(t)| \leqslant b\}$.*

COROLLARY 1. *In addition to hypotheses* H_1–H_5, *we assume* $f(t)$ *is* ω*-periodic and* $a(M) < a_0$, $b(M) < b_0$. *Then the bounded solution of* (1), *whose existence is ensured by Theorem* 2, *is also* ω*-periodic.*

THEOREM 4. *In addition to assumptions* H_1–H_5 *we suppose* $f(\cdot)$ *is* S^2*-almost periodic, i.e., the* $L^2([0, 1])$*-valued function*

$$\tilde{f}(t); R \to f(t + \cdot) \in L^2([0, 1])$$

is almost periodic. Then, if $a(M) < a_0$ *and* $b(M) < b_0$, *the bounded solution* $x(t)$ *of* (1) *is also almost periodic with respect to the norm* $\| x \|_1 = \sup_{t\in R} | x(t)|_1$, *where* $| x(t)|_1 = | x(t)| + | \dot{x}(t)|$.

THEOREM 5. *In addition to assumptions* H_1, H_2, *we suppose* $J_0(x) \leqslant K_3(r_0) | x |^{2-\alpha}$, $0 \leqslant \alpha \leqslant p_1$. *Then any bounded solution* $x(t)$ *on* $[0, \infty)$ *of* (1) *has the following decay property*:

(i) *If* $\delta(t) = (\int_t^{t+1} | f(s)|^{(p_0+2)/(p_0+1)} ds)^{(p_0+1)/(p_0+2)} \to 0$ *as* $t \to \infty$, *then* $\lim_{t\to\infty} | x(t)| = \lim_{t\to\infty} | \dot{x}(t)| = 0$.

(ii) *If* $\eta = (p_0 + 2)\{(p_1 + 2)^2 - 2 - \alpha\}/(2 + \alpha)(p_1 + 2) - 1 > 0$ *and* $\delta(t) \leqslant K | t |^{-(\theta+1)(p_0+1)/(p_0+2)}$ *with* $\exists\theta > 1/\eta$, $\exists K > 0$, *then* $V(x(t)) \leqslant Ct^{-1/\eta}$ *for* $t > 0$.

(iii) *If* $\eta = 0$, *i.e.,* $p_0 = p_1 = \alpha = 0$ *and* $\delta(t) \leqslant Ke^{-\theta t}$, $K > 0$, $\theta > 0$, *then* $V(x(t)) \leqslant Ce^{-\theta_1 t}$ *with some* $\theta_1 > 0$. *In* (ii) *and* (iii), *C denotes some constants.*

THEOREM 6. *Let us assume* H_3, H_4 *and let* $x_i(t)$ $(i = 1, 2)$ *be a bounded solution on* $[0, \infty)$ *of Eq.* (1) *with* $f(t)$ *replaced by* $f_i(t)$ $(i = 1, 2)$. *Then, if* $x_i(t) \in B_1^{a,b}$, $a < a_0$, $b < b_0$, *we have*:

(i) *If* $\delta(t) = (\int_t^{t+1} | f_1(s) - f_2(s)|^2 ds)^{1/2} \to 0$ *as* $t \to \infty$, *then* $\lim_{t\to\infty} | x_1(t) - x_2(t)|_1 = 0$.

(ii) *If* $\delta(t) \leqslant Ke^{-\theta t}$, K, $\theta > 0$, *then* $| x_1(t) - x_2(t)|_1 \leqslant Ce^{-\theta_1 t}$ *with some* C, $\theta_1 > 0$.

Here we shall give simple examples.

EXAMPLE 1. Consider the scalar-valued equation

$$\ddot{x}(t) + | \dot{x}(t)|^{p_0} \dot{x}(t) + | x(t)|^{p_1} x(t) = f(t), \qquad p_0, p_1 \geqslant 0.$$

In this case hypotheses H_1, H_2 are satisfied for any r_0, $r_1 > 0$, and consequently Theorems 1, 2, and 5 are applied to this equation.

EXAMPLE 2. Consider also the scalar-valued equation

$$\ddot{x}(t) + (1 + x(t)^2)(\dot{x}(t) + \dot{x}(t)^2) + x(t) + x(t)^2 = f(t).$$

In this case it is easy to see that hypotheses H_1–H_4 are satisfied with $p_0 = p_1 = 0$ and with $r_0, r_1 < 1$.

Finally in this section we give two lemmas concerning the solutions of (1), whose proofs are standard and omitted.

LEMMA 1. *Let $x(t)$ be a solution of* (8) *on an interval* $[T_1, T_2]$. *Then we have*

$$\begin{aligned}(\dot{x}(t_2), \phi(t_2)) - (\dot{x}(t_1), \phi(t_1)) - \int_{t_1}^{t_2} (\dot{x}(t), \dot{\phi}(t))\, dt \\ + \int_{t_1}^{t_2} (\rho(t, x(t), \dot{x}(t)) + \beta(x(t)) - f(t), \phi(t))\, dt = 0\end{aligned} \tag{12}$$

for $\forall t_1, \forall t_2 \in [T_1, T_2]$ *and for* $\forall \phi \in C^1([T_1, T_2])$.

LEMMA 2. *Let $x(t)$ be as in Lemma* 1. *Then we have*

$$V(x(t_2)) - V(x(t_1)) + \int_{t_1}^{t_2} (\rho(t, x(t), \dot{x}(t))\, dt = \int_{t_1}^{t_2} (f(t), \dot{x}(t))\, dt \tag{13}$$

for $\forall t_1, \forall t_2 \in [T_1, T_2]$.

3. PROOFS OF THEOREMS 1 AND 5

Let $x(t)$ be a local solution of (8) on an interval, say $[0, T]$, with initial conditions $x(0) = x_0$, $x(0) = x_1$. Such a solution exists by a standard theory [4]. In order to prove Theorem 1, it suffices to show $|x(t)| < r_0$ and $|x(t)| < (2D_0)^{1/2}$ for $\forall t \in [0, T]$. First we shall show T may be chosen as $T > 1$.

Indeed, taking $t_1 = 0$ and $t_2 = t$ with $0 < t \leqslant \min(1, T)$ in (13), we obtain

$$V(x(t)) + \int_0^t (\rho(s, x(s), \dot{x}(s)), \dot{x}(s))\, ds = V(x(0)) + \int_0^t (f(s), \dot{x}(s))\, ds$$

and, using assumption H_1, Young's inequality,

$$V(x(t)) + K_0 \int_0^t |\dot{x}(s)|^{p_0+2}\, ds \leqslant V(x(0)) + C_0(M) + K_0 \int_0^t |\dot{x}(s)|^{p_0+2}\, ds,$$

where

$$C_0(M) = (p_0 + 1)/(p_0 + 2) \cdot \{(p_0 + 2) K_0\}^{-1/(p_0+1)} \cdot M^{(p_0+2)/(p_0+1)}.$$

Hence

$$V(x(t)) \leqslant V(x(0)) + C_0(M) = \|(x_0, x_1)\|_V + C_0(M). \tag{14}$$

Therefore if we choose M_1 so that

$$C_0(M_1) + V(x(0)) = D_0 , \tag{15}$$

which is possible because of the form of $C_0(M)$ and the assumption $V(x(0)) = \|(x_0 , x_1)\|_V < D_0$, we have, for $M < M_1$,

$$V(x(t)) < D_0 \tag{16}$$

for $0 \leqslant t \leqslant \min(1, T)$.

The inequality (16) together with the assumption $| x_0 | < \overline{r_0}$ implies

$$| x(t)| < \overline{r_0} \leqslant r_0 \qquad \text{and} \qquad | \dot{x}(t)| < r_1 \tag{17}$$

for $0 \leqslant t \leqslant \min(1, T)$, from which we can conclude that we may assume $T > 1$ and (17) is valid for $0 < t \leqslant 1$.

Next we shall show that $x(t)$ can be continued to $[0, +\infty)$ and (17) holds on $[0, +\infty)$. For this it suffices again to see (16) is valid as long as $x(t)$ exists. Suppose that our assertion is false. Then there exists a time T (>1) such that

$$V(x(T)) = D_0 \tag{18}$$

and

$$V(x(t)) < D_0 \qquad \text{for} \quad t < T. \tag{19}$$

In (13) we take $t_2 = T$ and $t_1 = T - 1$ to obtain

$$\begin{aligned} V(x(T)) &+ \int_{T-1}^{T} (\rho(s, x(s), \dot{x}(s)), \dot{x}(s))\, ds \\ &= V(x(T-1)) + \int_{T-1}^{T} (f(s), \dot{x}(s))\, ds. \end{aligned} \tag{20}$$

From (20), it follows that

$$\int_{T-1}^{T} (\rho(s, x(s), \dot{x}(s)), \dot{x}(s)) \leqslant \int_{T-1}^{T} (f(s), \dot{x}(s))\, ds$$

and

$$\int_{T-1}^{T} | \dot{x}(s)|^{p_0+2}\, ds \leqslant \left(\frac{1}{K_0} M\right)^{(p_0+2)/(p_0+1)}. \tag{21}$$

Here we used assumption H_1 and Hölder's inequality. The integral inequality (21) implies that there exist times $t_1 \in [T - 1, T - \frac{3}{4}]$ and $t_2 \in [T - \frac{1}{4} , T]$ such that

$$| \dot{x}(t_i)| \leqslant 4^{1/(p_0+2)}(M/K_0)^{1/(p_0+1)} \tag{22}$$

for $i = 1, 2$.

On the other hand, taking $\phi = x$ in (12) and integrating over $[t_1, t_2]$ we obtain

$$\int_{t_1}^{t_2} J_1(x) = \int_{t_1}^{t_2} (\beta(x(s)), x(s))\, ds$$
$$\leqslant |(\dot{x}(t_2), x(t_2))| + |(\dot{x}(t_1), x(t_1))|$$
$$+ \int_{t_1}^{t_2} \{| \dot{x}(s)|^2 + (\rho(s, x(s), \dot{x}(s)), x(s)) + |(f(s), x(s))|\}\, ds$$

and by (2), (3), (21), (22),

$$K_2 \int_{t_1}^{t_2} | x(s)|^{p_1+2}\, ds \leqslant 2 \cdot 4^{1/(p_0+2)} \left(\frac{M}{K_0}\right)^{1/(p_0+1)} r_0 + \left(\frac{M}{K_0}\right)^{2/(p_0+1)}$$
$$+ K_1(r_0, r_1) \int_{t_1}^{t_2} | \dot{x}(s)|\, | x(s)|\, ds + \int_{t_1}^{t_2} |(f(s), x(s))|\, ds$$
$$\leqslant K_2 C_2(M, r_0, r_1), \tag{23}$$

where

$$C_2(M, r_0, r_1) = \frac{1}{K_2} \{2 \cdot 4^{1/(p_0+2)} r_0 (M/K_0)^{1/(p_0+1)} + M r_0$$
$$+ K_1 r_0 (M/K_0)^{1/(p_0+1)} + (M/K_0)^{2/(p_0+1)}\}.$$

From (21) and (23) we can conclude that there exists a time $t^* \in [t_1, t_2]$ such that

$$| x(t^*)|_1 \leqslant 2(M/K_0)^{1/(p_0+1)} + 2C_2(M, r_0, r_1)^{1/(p_1+2)}. \tag{24}$$

Since $V(x(t)) \to 0$ as $| x(t)|_1 \to 0$, we have from (24) $V(x(t^*)) \leqslant C_3(M, r_0, r_1)$, where $C_3(M, r_0, r_1)$ is a certain constant tending to 0 as $M \to 0$.

Now we are ready to derive a contradiction. In (13) we take $t_2 = T$ and $t_1 = t^*$ to obtain

$$D_0 - C_3(M, r_0, r_1) + \int_{t^*}^{T} (\rho(s, x(s), \dot{x}(s)), \dot{x}(s))\, ds \leqslant \int_{t^*}^{T} (f(s), \dot{x}(s))\, ds$$

and hence

$$D_0 \leqslant C_3(M, r_0, r_1) + C_0(M). \tag{25}$$

Therefore if we choose M_2 as the smallest number such that $C_3(M_2) + C_0(M_2) = D_0$, and moreover if we assume $M < M_2$, inequality (25) yields a contradiction. Consequently if we put $M_0 = \min(M_1, M_2)$, the proof of Theorem 1 is completed.

Remark 1. As is easily seen from the proof, we can obtain, in fact, instead of (16),

$$V(x(t)) \leqslant \max(C_3(M) + C_0(M), \|(x_0, x_1)\|_V + C_0(M)). \tag{26}$$

Next, we shall prove Theorem 5. Let $x(t)$ be any bounded solution of (1) on $[0, \infty)$ with bounds $|x(t)| \leqslant r_0$ and $|\dot{x}(t)| \leqslant r_1$. Then in a similar manner deriving inequality (21) we have

$$\begin{aligned} \int_{t-1}^{t} |\dot{x}(s)|^{p_0+2}\, ds \leqslant 2K_0^{-1}(V(x(t-1)) &- V(x(t)) \\ &+ (2K_0^{-1})^{(p_0+2)/(p_0+1)}\, \delta(t)^{(p_0+2)/(p_0+1)} \\ (\equiv A(t)^{p_0+2}) \end{aligned} \tag{27}$$

for $\forall t\ (\geqslant 1)$. Thus we can obtain as in (23)

$$K_2 \int_{t_1}^{t_2} |x(s)|^{p_1+2}\, ds \leqslant (2 \cdot 4^{1/(p_0+2)} + K_1) \max_{s\in[t-1,t]} |x(s)|\, A(t) + A(t)^2 \tag{28}$$

for some $t_1 \in [t-1, t-\frac{3}{4}]$, $t_2 \in [t-\frac{1}{4}, t]$.

From (27) and (28) it follows that there exists a point $t^* \in [t-1, t]$ such that

$$\begin{aligned} |\dot{x}(t^*)|^2 + |x(t^*)|^{2+\alpha} \leqslant A(t)^2 + C_4(r_0, r_1)\Big(&\max_{s\in[t-1,t]} V(x(s))^{1/(p_1+2)} A(t) \\ &+ A(t)^2\Big)^{(2+\alpha)/(p_1+2)} \end{aligned}$$

and hence, by the assumption $J_0(x) \leqslant K_3(r_0)\, |x|^{2+\alpha}$,

$$\begin{aligned} &\max_{s\in[t-1,t]} V(x(s)) \\ &\leqslant V(x(t^*)) + \int_{t-1}^{t} (\rho(s, x(s), \dot{x}(s)), \dot{x}(s))\, ds + \int_{t-1}^{t} (f(s), \dot{x}(s))\, ds \\ &\leqslant C_5(r_0, r_1)\{A(t)^2 + A(t)^{p_0+2} + \max_{s\in[t-1,t]} V(x(s))^{(2+\alpha)/(p_1+2)^2} A(t)^{(2+\alpha)/(p_1+2)} \\ &\qquad + A(t)^{(4+2\alpha)/(p_1+2)}\} + \delta(t)^{(p_0+2)/(p_0+1)} \\ &\leqslant C_6(r_0, r_1)\, A(t)^{(2+\alpha)(p_1+2)/\{(p_1+2)^2-2-\alpha\}} + \delta(t)^{(p_0+2)/(p_0+1)}. \end{aligned} \tag{29}$$

In the above we have used the fact

$$(2+\alpha)(p_1+2)/\{(p_1+2)^2 - 2 - \alpha\} \leqslant (4+2\alpha)/(p_1+2) \leqslant 2.$$

From (29) and the definition of $A(t)$ of (27) we obtain

$$\max_{s\in[t-1,t]} V(x(s))^{1+\eta} \leqslant C_7(r_0, r_1)\{V(x(t-1)) - V(x(t)) + \delta(t)^{(p_0+2)/(p_0+1)}\} \tag{30}$$

where we recall $\eta \equiv (p_0+2)\{(p_1+2)^2 - 2 - \alpha\}/(p_1+2)(2+\alpha) - 1 \geqslant 0$.

Now applying the following lemma to (30) we obtain Theorem 5 immediately.

LEMMA 3. *Suppose that $\phi(t)$ be a bounded nonnegative function on R^+, satisfying*

$$\max_{s\in[t,t+1]} \phi(s)^{1+\eta} \leqslant K_5(\phi(t) - \phi(t+1)) + g(t)$$

for $t \geqslant 0$, where K_5 is a positive constant, $g(t)$ a nonnegative function, η a nonnegative constant. Then we have:

(i) *If $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} \phi(t) = 0$.*

(ii) *If $\eta > 0$, $g(t) \leqslant K_6 \mid t \mid^{-\theta-1}$ with $\exists\theta > 1/\eta$, $\exists K_6 > 0$, then $\phi(t) \leqslant Ct^{-1/\eta}$ for $t > 0$.*

(iii) *If $\eta = 0$, $g(t) \leqslant K_7 e^{-\theta t}$ with $\exists\theta > 0$, $\exists K_7 > 0$, then we have $\phi(t) \leqslant Ce^{-\theta_1 t}$ with $\theta_1 = \min(\theta, \log(K_7/(K_7 - 1)))$.*

(C denotes some positive constants.)

For a proof of Lemma 3, see [8].

4. PROOF OF THEOREM 2

On the basis of Theorem 1, we shall prove Theorem 2. For this purpose consider the initial value problem:

$$\ddot{x}(t) + \rho(t, x(t), \dot{x}(t)) + \beta(x(t)) = f(t), \qquad -r \leqslant t < +\infty,$$

with

$$\dot{x}(-r) = x(-r) = 0, \qquad r = 1, 2,\ldots .$$

Then, by Theorem 1, a bounded solution $x_r(t)$ on $[-r, +\infty)$ exists if $M < \overline{M_0}$. In particular we know (Remark 1)

$$x_r(t) \in W \qquad \text{and} \qquad V(x_r(t)) \leqslant C_3(M) + C_0(M) < D_0 \tag{31}$$

and hence

$$|\, x_r(t)| \leqslant a(M) < r_0 \qquad \text{and} \qquad |\, \dot{x}_r(t)| \leqslant b(M) < r_1 \tag{32}$$

for certain constants $a(M)$, $b(M)$, tending to 0 as $M \to 0$.

From (32) and Eq. (8) it is easy to see that x_r belongs to

$$C^{(p_0+1)/(p_0+2)}([-r, \infty]) \qquad \text{and} \qquad |\, x_r(t)|_{C^{(p_0+1)/(p_0+2)}} \leqslant C_8(M) < +\infty.$$

Since a bounded set in $C^{1+\alpha}([-T, T])$, $T > 0$, $\alpha > 0$, is compact in $C^1([-T, T])$, we can extract a subsequence from $\{x_r(t)\}$, which will be denoted by the same symbol for simplicity, converging to a function $x(t) \in C^1((-\infty, \infty))$ as $r \to \infty$ in the following sense:

$$x_r(t) \to x(t)\ (r \geqslant T) \text{ uniformly on } [-T, T] \text{ for } \forall T > 0$$

with respect to the norm $|\cdot|_1$.

Therefore, letting $r \to \infty$ in (8) with x replaced by x_r, we obtain (8) for the above $x(t)$ and for $\forall t_1, \forall t_2 \in R$. Of course we see, by (32), $|x(t)| \leqslant a(M)$ and $|\dot{x}_r(t)| \leqslant b(M)$. Thus $x(t)$ is a required bounded solution.

5. Proofs of Theorems 3, 4, and 6 and Corollary 1

Let $x_i(t)$ be bounded solutions of system (1) with f replaced by $f_i(t)$, $i = 1, 2$. We assume

$$|x_i(t)| \leqslant a \leqslant r_0 \quad \text{and} \quad |\dot{x}_i(t)| \leqslant b \leqslant r_1, \quad i = 1, 2. \tag{33}$$

Putting $w(t) = x_1(t) - x_2(t)$, we can obtain, as in Lemma 2,

$$\begin{aligned} &\tfrac{1}{2} |w(t_2)|_E^2 - \tfrac{1}{2} |w(t_1)|_E^2 + \int_{t_1}^{t_2} (\rho(t, x_1(t), \dot{x}_1(t)) - \rho(t, x_2(t), \dot{x}_2(t)), \dot{w})\, dt \\ &\quad + \int_{t_1}^{t_2} (\tilde{\beta}(x_1(t)) - \tilde{\beta}(x_2(t)), \dot{w}(t))\, dt = \int_{t_1}^{t_2} (f_1(t) - f_2(t), \dot{w}(t))\, dt, \end{aligned} \tag{34}$$

where $|w(t)|_E^2 = |\dot{w}(t)|^2 + (Bw(t), w(t))$.

Using assumptions (4) and (6) we see from (34) that

$$\begin{aligned} &|w(t_2)|_E^2 - |w(t_1)|_E^2 + K_0 \int_{t_1}^{t_2} |\dot{w}(s)|^2\, ds \\ &\quad \leqslant 2(K_4(a) + K_3(a, b)) \int_{t_1}^{t_2} |w(s)|\, |\dot{w}(s)|\, ds + K_0^{-1} \delta_{t_1, t_2}, \end{aligned} \tag{35}$$

where we assume $\delta_{t_1, t_2} \equiv (\int_{t_1}^{t_2} |f_1(t) - f_2(t)|^2\, ds)^{1/2} < +\infty$, and hence

$$\int_t^{t+1} |\dot{w}(s)|^2\, ds \leqslant A(t)^2 \qquad \text{for} \quad \forall t \in R \quad (\text{or } R^+), \tag{36}$$

where we set

$$\begin{aligned} A(t)^2 \equiv K_0^{-1} \Big\{ &|w(t)|_E^2 - |w(t+1)|_E^2 + 2(K_4(a) + K_3(a, b)) \\ &+ \int_t^{t+1} |w(s)|\, |\dot{w}(s)|\, ds + K_0^{-1}\, \delta(t)^2 \Big\} \qquad (\delta_{t, t+1} \equiv \delta(t)). \end{aligned}$$

Thus, as in the proof of (23) we can obtain

$$(C_0 - K_4(a) - K_3(a, b)) \int_{t_1}^{t_2} | w(s)|^2 \, ds$$
$$\leqslant (4 + K_2(r_0, r_1)) \max_{s\in[t,t+1]} | w(s)| \, A(t) + A(t)^2 + \max_{s\in[t,t+1]} | w(s)| \, \delta(t) \tag{37}$$

for some $t_2 \in [t + \frac{3}{4}, t + 1]$, $t_1 \in [t, t + \frac{1}{4}]$.

Now we choose $a_1, b_1 > 0$ so that $C_0 - K_4(a) - K_3(a, b) > 0$ for $a \leqslant a_1$, $b \leqslant b_1$, which is possible by our assumptions. Hereafter we assume $a \leqslant a_1$ and $b \leqslant b_1$. Then inequalities (36), (37) imply that there exists a point $t^* \in [t_1, t_2]$ such that

$$\begin{aligned} | w(t^*)|_E^2 &\leqslant | \dot{w}(t^*)|^2 + C_1 | w(t^*)|^2 \\ &\leqslant 2A(t)^2 + 2C_1(C_0 - K_4(a) - K_3(a, b))^{-1} \\ &\quad \times \{(4 + K_2(r_0, r_1)) \max_{s\in[t,t+1]} | w(s)| \, A(t) + A(t)^2 + \max_{s\in[t,t+1]} | w(s)| \, \delta(t)\}. \end{aligned} \tag{38}$$

Also, we can again use (35) with $t_2 = s$, $t_1 = t^*$ (or $t_2 = t^*$, $t_1 = s$) to obtain easily

$$\begin{aligned} \max_{s\in[t,t+1]} | w(s)|_E^2 \leqslant{} & | w(t^*)|_E^2 + (1 + 2K_2(r_0, r_1)) \, A(t)^2 \\ & + (K_3(a, b) + K_4(a)) \max(1, C_0^{-1}) \max_{s\in[t,t+1]} | w(s)|_E^2 \\ & + \delta(t)^2. \end{aligned} \tag{39}$$

Combining (38) and (39), and recalling the definition of $A(t)$, an easy calculation yields

$$\begin{aligned} \max_{s\in[t,t+1]} | w(s)|_E^2 \leqslant{} & C_9(r_0, r_1)(| w(t)|_E^2 - | w(t + 1)|_E^2) \\ & + C_{10}(a, b) \max_{s\in[t,t+1]} | w(s)|_E^2 + C_{11}\delta(t)^2, \end{aligned} \tag{40}$$

where the constants are given by

$$\begin{aligned} C_9(r_0, r_1) = K_0^{-1}\{ & 3 + 2K_2 + 4C_1^2(4 + K_2)^2 (C_0 - K_4(a) - K_3(a, b))^{-2} \\ & + 2C_1(C_0 - K_4(a) - K_3(a, b))^{-1}\}, \end{aligned}$$

$$C_{10}(a, b) = \tfrac{1}{2} + (1 + K_0^{-1})(K_3(a, b) + K_4(a)) \max(1, C_0^{-1}),$$

and

$$C_{11} = (1 + K_0^{-2} + 2C_1^2(C_0 - K_4 - K_3)^{-2}).$$

Noting that $K_3(a, b) + K_4(a) \to 0$ as $a + b \to 0$, we can choose positive constants $a_0(<a_1)$ and $b_0(<b_1)$ such that

$$(1 + K_0^{-1})(K_3(a, b) + K_4(a)) \max(1, C_0^{-1}) < \tfrac{1}{2}$$

for $0 < \forall a < a_0$, $0 < \forall b < b_0$.

Thus, for such a, b, we have from (40)

$$\max_{s \in [t, t+1]} | w(s)|_E^2 \leqslant C_9'(| w(t)|_E^2 - | w(t + 1)|_E^2) + C_{11}' \delta(t)^2, \tag{41}$$

where

$$C_9' = (1 - C_{10})^{-1} C_9 \qquad \text{and} \qquad C_{11}' = (1 - C_{10})^{-1} C_{11}.$$

Combining lemma 3 with estimate (41) for $t \in R^+$ yields Theorem 6. In particular, if $f_1 = f_2$, we have

$$\max_{s \in [t, t+1]} | w(s)|_E^2 \leqslant C_9'(| w(t)|_E^2 - | w(t + 1)|_E^2) \qquad \text{for} \quad t \in R,$$

which easily implies $w(t) \equiv 0$. Thus Theorem 3 is proved. Corollary 1 follows from Theorem 3 immediately. Finally, we shall prove Theorem 4. Let $\{t_n\}$ be any real sequence and consider the sequence of bounded solutions $\{u(t + t_n)\}$ of Eq. (1) with $f(t)$ replaced by $f(t + t_n)$. To prove Theorem 4 it suffices by Bochner's criterion to show that $\{x(t + t_n)\}$ contains an sequence which is convergent uniformly on R with respect to the norm $| \cdot |_1$. Putting $w_{m,n}(t) = x(t + t_n) - x(t + t_m)$, we have from (41)

$$\max_{s \in [t, t+1]} | w_{m,n}(t)|^2 \leqslant C_9'(| w_{m,n}(t)|_E^2 - | w_{m,n}(t + 1)|_E^2) + C_{11}'\delta_{(m,n)}^2 \quad \text{for} \quad t \in R, \tag{42}$$

where

$$\delta_{(m,n)} = \sup_{t \in R} \left(\int_t^{t+1} | f(s + t_n) - f(s + t_m)|^2 \, ds \right)^{1/2}.$$

By the almost periodicity assumption on $f(t)$, we may assume $\delta_{(m,n)} \to 0$ as $m, n \to \infty$. Thus we can conclude from (42) that $| w_{m,n}(t)|_E \to 0$ uniformly on R as $m, n \to \infty$ (cf. [6]), which proves Theorem 4.

Remark 2. Let us consider the special case: $\rho(t, x, \dot{x}) \equiv \rho(\dot{x})$ and $\beta(x) = B \cdot x$ (B: positive matrix). Suppose that $\rho(0) = 0$, $| \rho(v_1) - \rho(v_2)| \leqslant K_1(r_1) | v_1 - v_2 |$ and $K_0(r_1) | v_1 - v_2 |^{p_0+2} \leqslant (\rho(v_1) - \rho(v_2), v_1 - v_2)$ if $| v_1 |$, $| v_2 | \leqslant r_1$. Then, by the quite similar argument in the proof of Theorem 5, we can obtain instead of (41)

$$\max_{s \in [t, t+1]} | w(s)|_E^{p_0+2} \leqslant \text{const}(| w(t)|_E^2 - | w(t + 1)|_E^2) + \text{const}\, \delta_0(t)^{(p_0+2)/(p_0+1)}, \tag{41$'$}$$

where

$$\delta_0(t) = \left(\int_t^{t+1} |f_1(s) - f_2(s)|^{(p_0+2)/(p_0+1)}\, ds\right)^{(p_0+1)/(p_0+2)}.$$

From (41)′ we can conclude, if $\delta_0(t)$ tends to 0 sufficiently rapidly as $t \to \infty$, that

$$|w(t)| + |\dot{w}(t)| \leqslant \text{const } t^{-1/p_0} (p_0 > 0), \qquad \text{or}, \qquad \text{const } e^{-\theta t} \qquad (p_0 = 0)$$

for some $\theta > 0$.

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